

Variational Principle of Hydrodynamics and Quantization by Stochastic Process

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The well-known hydrodynamical representation of the Schrödinger equation is reformulated by extending the idea of Nelson-Yasue's stochastic variational method. The fluid flow is composed by the two stochastic processes from the past and the future, which are unified naturally by the principle of maximum entropy. We show that This formulation is easily applicable to the quantization of scalar fields.

I. INTRODUCTION

It is almost a century since the present form of quantum mechanics has been established. It is one of the most beautiful and sound theories from the mathematical point of view. Because of the recent development of experimental technology, we are now possible to perform observations related to fundamental aspects in quantum mechanics [1].

So far, there is no experimental observation that throws doubt on its validity, but we may encounter a situation where the framework of quantum mechanics will be forced to be modified in future. In fact, the unification of quantum mechanics and general relativity is still unknown. Therefore, it is important to deepen understandings of the foundation of quantum mechanics.

Of course such efforts have already been done from very early stage of the formulation of quantum mechanics, as de Broglie, Bohm and Vigier [2]. In 1966, Nelson [3] considered that the quantum fluctuation can be regarded as the Bernstein-type stochastic behavior of a particle trajectory and succeeded in deriving the Schrödinger equation [4]. Later, Yasue [5] [6] reformulated this idea in the form of optimization of an action for the stochastic variables, known as stochastic variational method (SVM). Several applications of Yasue's SVM have been found in Ref. [7, 8] and references therein.

In this paper, we propose an alternative representation of the stochastic quantization within the variational approach. In the present formulation, we first require the maximum of the entropy associated with stochastic trajectories in order to accomodate consistently

the forward and backward stochastic processes in the Nelson-Yasue approach. We then show that the action of the system can be expressed as that of a classical ideal fluid with the quantum correction as the internal energy of the fluid element [9]. The variation of our action can be cast into the form of the action principle of the well-known quantum mechanics if we change the variables adequately.

The present paper is organized as follows. In Sec. II, for the sake of book-keeping, the classical action principle for one particle system and its generalization to the statistical ensemble are summarized. In Sec.III, we introduce the two Brownian motions necessary in the variational formulation and postulate the maximum entropy principle. In Sec.IV, the action of this combined fluid is established. From this action, we show that it is always possible to construct the linear representation of the dynamics in term of the wave function ψ . In Sec. V, we discuss the correspondence to the momentum and Hamiltonian as the usual operator forms and the significance of eigenstates and eigenvalues through Noether's theorem. In Sec. VI, application of the present approach to the field quantization is discussed. Sec.VII is devoted to the summary and discussion.

II. CLASSICAL SINGLE-PARTICLE SYSTEM AND COMPLEX REPRESENTATION OF ITS ACTION

As is well-known, the usual classical Lagrangian of a one-particle system is given by

$$L_p \left(\mathbf{r}, \frac{d\mathbf{r}}{dt} \right) = \frac{m}{2} \left(\frac{d\mathbf{r}}{dt} \right)^2 - V(\mathbf{r}), \quad (1)$$

where \mathbf{r} and V are the particle trajectory and the potential, respectively. For the sake of later convenience, we express Lagrangian in terms of hydrodynamic variables.

Let us now consider a set of an infinite number of systems whose dynamics are equivalent each other and the initial and final conditions are specified as distribution functions of particles and velocities. That is, we are thinking of a gas of collisionless particles (dust) under the influence of the potential. In this ensemble, the dynamical variables are given by the particle distribution function $\rho(\mathbf{r}, t)$ and the velocity field, $\mathbf{v}(\mathbf{r}, t)$, instead of the particle trajectory $\mathbf{r}(t)$. This is the Euler-coordinate representation of one-particle system which is common in the argument of hydrodynamics. This Lagrangian density is given by

$$L_h = \frac{1}{2} m \rho \mathbf{v}^2 - \rho V + \kappa \lambda(\mathbf{r}, t) \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right], \quad (2)$$

where the last term represents the dynamical constraint associated with the conservation of the particle number,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (3)$$

and $\lambda(\mathbf{r}, t)$ is a field corresponding to the Lagrangian multiplier. Here we introduce a constant κ to make λ an adimensional quantity. Thus, κ has the dimension of \hbar .

The variations for \mathbf{v} and ρ lead to the following equation,

$$\frac{\partial}{\partial t} \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{m} \nabla V, \quad (4)$$

where

$$\mathbf{v}(\mathbf{x}, t) = \frac{\kappa}{m} \nabla \lambda(\mathbf{x}, t). \quad (5)$$

In the one-particle system, ρ is given by Dirac's delta function along the particle trajectory and then the above equation is reduced to Newton's equation of motion.

Eq.(4) can be interpreted as the Euler equation for a hypothetic "fluid" which consists of non-interacting particles. In fact, if we replace the potential V by U which contains the contribution of an internal energy, the potential gradient $\nabla V/m$ is replaced by the pressure gradient $\nabla P/\rho$ as is the case of the Euler equation.

We can reduce the number of variables of variation by using Eq. (5) in Eq. (2) as

$$I_h[\rho, \lambda] = \int_{t_I}^{t_F} dt \int d^3\mathbf{r} \rho \left\{ -\frac{\kappa^2}{2m} (\nabla \lambda)^2 - V - \kappa \dot{\lambda} \right\}, \quad (6)$$

where $t_I(t_F)$ denotes the initial (final) time. Note that the same can be derived even from the Hamilton-Jacob theory, where λ plays the role of generator S [?]. Here, the last term on the left hand side appears as the influence of the constraint condition associated with the continuity equation of ρ .

Now, instead of the two scalar variables, ρ and λ , let us introduce a complex variable

$$\psi = \sqrt{\rho} e^{i\lambda}. \quad (7)$$

Then the above action is further re-expressed as

$$I_h(\rho, \lambda) = \int_{t_i}^{t_f} dt \int d^3\mathbf{r} \psi^* \left\{ i\kappa \frac{\partial}{\partial t} - \bar{H} \right\} \psi, \quad (8)$$

where

$$\bar{H} = -\frac{\kappa^2}{2m} \nabla^2 + \bar{V},$$

with the potential

$$\bar{V} = V - \frac{\kappa^2}{2m} (\nabla \ln \sqrt{\rho})^2. \quad (9)$$

In short, the classical action (6) can be cast into a similar form of the quantum mechanical action [2]. Of course, they are still different because of the ρ -dependent term in \bar{V} . However, if the potential V contained an additional term, such as the internal energy of fluid, and if this term cancels out the second term in Eq.(9), the corresponding classical action would be the same as that of quantum mechanics.

In the following, we show that the above scenario can in fact be constructed.

III. BROWNIAN MOTION

So far, we considered that the particle trajectory is deterministic, but let us now suppose that it is stochastic due to some unknown external factor. In classical fluids, such a fluctuation occurs as the influence of the nature of fluid elements which consist of many internal degrees of freedom [7]. On the other hand, the origin of the stochasticity in quantum mechanics is not known. Here, we will not discuss what is this origin, but show that such a stochasticity can generate the internal energy.

In order to introduce the stochasticity, we suppose that the particle trajectory obeys a Brownian motion, which is characterized by

$$d\mathbf{r} = \mathbf{u}_F(\mathbf{r}, t) dt + \vec{\xi}_F, \quad dt > 0. \quad (10)$$

where $\vec{\xi}_F$ is the Gaussian noise defined by the probability distribution function,

$$P(\xi) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\xi^2}{2\sigma^2}}, \quad (11)$$

where $\sigma^2 = \nu |dt|$ with ν is a constant parameter which characterizes the intensity of the noise and has a dimension of L^2/T . We refer Eq.(10) as forward stochastic difference equation (FSDE).

The particle distribution of this Brownian motion is calculated from FSDE and given by the following Fokker-Planck equation,

$$\frac{\partial \rho_F}{\partial t} + \nabla \cdot \mathbf{j}_F = 0, \quad (12)$$

where

$$\rho_F(\mathbf{r}, t) = \langle \delta(\mathbf{r} - \mathbf{r}(t)) \rangle_F, \quad (13)$$

$$\mathbf{j}_F(\mathbf{r}, t) = \rho_F(\mathbf{u}_F - \nu \nabla \ln \rho_F). \quad (14)$$

Here $\langle O(t) \rangle_F$ represents the average of O at the instant t over the whole events of the ensemble satisfying Eq(10).

This equation, for a given \mathbf{u}_F , could be solved forward in time as an initial value problem. However, we need to fix not only the initial but also the final particle distributions in the variational approach. The solution of the above Fokker-Planck equation does not satisfy a given final distribution in general.

One possible way to control the final distribution, we think of a stochastic process backward in time. That is, instead of FSDE (10), we consider the backward stochastic differential equation (BSDE),

$$d\mathbf{r} = \mathbf{u}_B(\mathbf{r}, t) dt + \vec{\xi}_B, \quad dt < 0. \quad (15)$$

where $\vec{\xi}_B$ is again the Gaussian noise as before obeying the probability distribution function, Eq.(11). The corresponding Fokker-Planck equation is

$$\frac{\partial \rho_B}{\partial t} + \nabla \cdot \mathbf{j}_B = 0, \quad (16)$$

where

$$\mathbf{j}_B = \rho_B(\mathbf{u}_B + \nu \nabla \ln \rho_B). \quad (17)$$

Note that the sign of the second term is opposite to Eq. (14).

We should construct a set of stochastic processes which satisfies the both initial and final conditions by using these two Brownian motions. For this, first let us consider trajectories which obey FSDE with a given initial condition and pass in the vicinity of \mathbf{r} at a certain time t which satisfies $t_I < t < t_F$. The number of the trajectories should be proportional to $\rho_F(\mathbf{r}, t)$. Similarly, the number of the trajectories which pass the same domain starting from the final distribution should be proportional to $\rho_B(\mathbf{r}, t)$. Therefore, the number N of the ways to construct a trajectory which combines these two should be proportional to the product of the densities $N \propto \rho_F(\mathbf{r}, t) \rho_B(\mathbf{r}, t)$.

Now we require that *the law of Nature describes the situation where this combined number is maximal for any instant t* . In other words, the entropy associated with this combination

of trajectories should be maximum. We then define the entropy

$$S[\rho_F, \rho_B] \equiv \int d^3\mathbf{r} N \ln N.$$

From the variation of this entropy $\delta S = 0$ with the following constraint conditions,

$$\int d^3\mathbf{r} \rho_F(\mathbf{r}, t) = \int d^3\mathbf{r} \rho_B(\mathbf{r}, t) = 1, \quad (18)$$

we obtain

$$\rho_F(\mathbf{r}, t) = \rho_B(\mathbf{r}, t). \quad (19)$$

Therefore the density of trajectories ρ which satisfy the two boundary conditions is given by

$$\rho(\mathbf{r}, t) = \rho_F(\mathbf{r}, t) = \rho_B(\mathbf{r}, t). \quad (20)$$

Once we establish Eq.(19), we get from Eqs.(12,16)

$$\rho(\mathbf{u}_F - \nu \nabla \ln \rho) - \rho(\mathbf{u}_B + \nu \nabla \ln \rho) = \nabla \chi + \nabla \times \mathbf{A}, \quad (21)$$

where \mathbf{A} is an arbitrary time-dependent vector field [7] and χ is a scalar function which satisfies $\nabla^2 \chi = 0$. In the presence of \mathbf{A} , the velocity field should contain a vortex which is related to a singularity in space. Similarly, since $\lim_{|\mathbf{r}| \rightarrow \infty} \nabla \chi \rightarrow 0$, $\chi = \text{const.}$, if χ has no singularity. Here, for simplicity, we assume $\nabla \chi = \mathbf{A} = 0$. Then we obtain

$$\mathbf{u}_F - \mathbf{u}_B = 2\nu \nabla \ln \rho. \quad (22)$$

This is nothing but the same relation called the consistency condition in SVM [5, 7].

Because of this condition, the two Fokker-Planck equations are reduced to the simple continuity equation,

$$\partial_t \rho + \nabla(\rho \mathbf{u}_T) = 0, \quad (23)$$

where

$$\mathbf{u}_T = \frac{\mathbf{u}_F + \mathbf{u}_B}{2}. \quad (24)$$

IV. ACTION PRINCIPLE AND QUANTUM MECHANICS

In the previous section, we introduce the stochastic process which is characterized by the two Brownian motions. As is in the hydrodynamic variational approach, the properties of

these Brownian motions can be incorporated as the constraint condition in the variational approach. That is, once we know the kinetic term T and potential term U , the action which we should optimize is given by

$$I_q = \int dt \int d^3\mathbf{r} \rho \left\{ T - U - \kappa \dot{\lambda} - \kappa \nabla \lambda \cdot \mathbf{u}_T \right\}.$$

Here the Fokker-Planck equations (23) is taken into account as the constraint conditions with a Lagrange multipliers, λ .

From Eq.(23), we can identify the velocity \mathbf{u}_T with that of the translational motion of the fluid. Therefore, the translational kinetic energy T should be

$$T = \frac{m}{2} \mathbf{u}_T^2. \quad (25)$$

On the other hand, the relative velocity,

$$\mathbf{u}_r = \mathbf{u}_F - \mathbf{u}_B, \quad (26)$$

is irrelevant to the translational motion of the fluid element but should be associated with its internal energy. Such a situation occurs in the case of the kinetic derivation of hydrodynamics, where the momentum of microscopic constituent particles are separated into two parts; one is to the fluid velocity and the other to the internal energy. Following this idea, the potential term is expressed as

$$U = \frac{m_{eff}}{2} \mathbf{u}_r^2 + V = 8\alpha^2 \nu^2 m (\nabla \ln \sqrt{\rho})^2 + V,$$

where $m_{eff} = \alpha^2 m$ is an effective mass, and α is still undetermined constant. As before, V is the external potential.

By using these results, we arrive at the following expression,

$$I_q [\rho, \lambda] = \int dt \int d^3\vec{r} \rho \left\{ -\frac{1}{2m} (\kappa \nabla \lambda)^2 - U - \kappa \dot{\lambda} \right\}. \quad (27)$$

Since κ is an arbitrary constant, we can always choose it so as to satisfy

$$\kappa = 4\alpha \nu m. \quad (28)$$

Therefore, as was done in Sec. II, this can be re-expressed in terms of the usual wave function as

$$I_q [\rho, \lambda] = \int dt \int d^3\vec{r} \psi^* (\mathbf{r}, t) [i\kappa \partial_t - \bar{H}] \psi (\mathbf{r}, t), \quad (29)$$

where the wave function is defined by Eq. (7) and

$$\bar{H} = -\frac{\kappa^2}{2m}\nabla^2 + V(\mathbf{r}). \quad (30)$$

Then the variation for ρ and λ of the action (29) leads to

$$\left[i\kappa\partial_t - \left(-\frac{\kappa^2}{2m}\nabla^2 + V(\mathbf{r}) \right) \right] \psi(\mathbf{r}, t) = 0, \quad (31)$$

If the parameter κ is identified with \hbar , this is nothing but the Schrödinger equation.

It should be noted that our result of the variation becomes non-linear if the parameter does not satisfy Eq. (28) [10]. However, because of the ambiguity for the definition of the phase λ , we can always find a parameter set satisfying Eq. (28) and thus a non-linear dynamics can be cast into the form of the linear equation (see also [11]). In other words, it seems that the wave function is a convenient representation but is not necessarily the fundamental element to construct quantum mechanics. This will be discussed in Sec. VII. In the following, we will refer this linear representation as ψ -representation.

V. DEFINITIONS OF PHYSICAL OPERATORS AND EIGENVALUES

In the canonical quantization, the momentum operator is defined so as to maintain the correspondence principle between the Poisson brackets and commutators. In the present formulation, it is defined through the Noether theorem.

The momentum is a conserved quantity associated with the invariance for the spatial translation. Suppose that the action Eq.(29) is invariant under the spatial translation. Then, the corresponding conserved Noether charge is

$$\mathbf{P} = \int d^3\mathbf{x} \rho \nabla \lambda. \quad (32)$$

That is, the event average of $\nabla \lambda$ is a conserved quantity. Similarly, the invariance for the time translation leads to the Noether conserved charge as

$$H = \int d^3\mathbf{x} \rho \left[\frac{m}{2}(\mathbf{u}_m^2 + \alpha^2 \nu^2 (\nabla \ln \rho)^2) + V \right]. \quad (33)$$

In ψ -representation, these conserved charges can be expressed as

$$\mathbf{P} = \int d^3\tilde{\mathbf{r}} \psi^*(\mathbf{r}, t) \frac{\hbar}{i} \nabla \psi(\mathbf{r}, t) \equiv \langle \hat{\mathbf{P}} \rangle, \quad (34)$$

$$H = \int d^3\tilde{\mathbf{r}} \psi^*(\mathbf{r}, t) \left(-\frac{\hbar^2}{2m} \nabla^2 + V \right) \psi(\mathbf{r}, t) \equiv \langle \bar{H} \rangle. \quad (35)$$

These are usual operator representations of physical quantities in quantum mechanics. For other observables related to the generator of some kind of transformation, for example, angular momentum can also be defined in a similar way.

The eigenvalues and corresponding eigenstates can be introduced following the classical arguments of von Neumann [12]. Let us consider an wave function ψ_P representing the corresponding expectation value by $\langle \rangle_P$. Suppose that this state satisfies

$$\langle (\hat{\mathbf{P}} - \langle \hat{\mathbf{P}} \rangle_P)^2 \rangle_P = 0. \quad (36)$$

Then it is clear that ψ_P is the solution of the following equation,

$$(\hat{\mathbf{P}} - \mathbf{p}_0) \psi_P(\mathbf{r}, t) = 0. \quad (37)$$

Here \mathbf{p}_0 is an eigenvalue. By the same argument, we can construct any eigenvalues and eigenstates of physical observables defined by the Noether theorem.

VI. APPLICATION TO FIELD QUANTIZATION

The stochastic quantization procedure in terms of variational principle in the previous sections can be extended in a straightforward way for any system described by a vector variable, say $\vec{\phi}$ in stead of \mathbf{r} . As an example, we sketch in below how the present formulation can be applied to a scalar field theory.

To apply our formulation for a scalar field system, we introduce the spatial lattice representation. In this representation, we can assign the field configuration $\phi(x)$ to a vector,

$$\phi(x) \rightarrow \vec{\phi}, \quad (38)$$

in such a way that the scalar product of two functions

$$(f, h) = \int d^3x f(x) h(x) \rightarrow \Delta^3\mathbf{x} \vec{f} \cdot \vec{h}$$

with $\Delta^3\mathbf{x}$ being the lattice volume.

In this representation, the classical action for the scalar field $\phi(x)$ can be written as

$$I_{c.f.} = \int_{t_i}^{t_f} dt \left\{ \frac{1}{2c^2} \Delta^3\mathbf{x} \left(\frac{d\vec{\phi}}{dt} \right)^2 + \frac{1}{2} \Delta^3\mathbf{x} \vec{\phi} \cdot \Delta_{\mathbf{x}} \vec{\phi} - V(\vec{\phi}) \right\}, \quad (39)$$

where V is the potential containing the mass term and $\Delta_{\mathbf{x}}$ is the matrix corresponding to the discretized Laplacian operator [8]. By denoting a formal correspondence

$$\vec{r} \rightarrow \vec{\phi}, \quad (40)$$

$$d^3\vec{r} \rightarrow d^{N^3}\vec{\phi} \equiv [D\phi], \quad (41)$$

we can repeat the analogous procedure in the previous sections and we have finally

$$I_{q.f.} = \int_{t_i}^{t_f} dt \int [D\phi] \rho[\phi] \left\{ \int d^3\mathbf{x} \left(\frac{(\kappa c)^2}{2} \left\{ \left(\frac{\delta \lambda}{\delta \phi(x)} \right)^2 - \left(\frac{\delta \ln \sqrt{\rho}}{\delta \phi(x)} \right)^2 \right\} + \frac{1}{2} \phi(x) \nabla^2 \phi(x) \right) - V[\phi] - \kappa \partial_t \lambda \right\}. \quad (42)$$

In the above, we used the notation in the continuum limit as

$$\frac{1}{\Delta^3\mathbf{x}} \nabla_{\vec{\phi}} \rightarrow \frac{\delta}{\delta \phi(x)}, \quad (43)$$

and $\rho[\phi]$ represents the functional distribution of field configuration ϕ .

As before, in terms of wave functional,

$$\Psi[\phi] \equiv \sqrt{\rho} e^{i\lambda}, \quad (44)$$

the above action is rewritten as

$$I_{q.f.} = \int_{t_i}^{t_f} dt \int [D\phi] \Psi^*[\phi] \left[i\kappa c \partial_{ct} - \hat{\mathcal{H}} \right] \Psi[\phi], \quad (45)$$

where

$$\hat{\mathcal{H}} = \int d^3\mathbf{x} \left\{ -\frac{\kappa^2 c^2}{2} \left(\frac{\delta}{\delta \phi(x)} \right)^2 - \frac{1}{2} (\nabla \phi(x))^2 \right\} + V. \quad (46)$$

Taking variations with respect to the two functionals $\rho[\phi]$ and $\lambda[\phi]$, we arrive at the functional Schrödinger equation

$$i \kappa c \partial_{ct} \Psi[\phi] = \hat{\mathcal{H}} \Psi[\phi].$$

Note that, for the scalar field system, the corresponding condition to Eq.(28) determines the relation between the noise intensity ν and the universal constant, κc .

VII. SUMMARY AND DISCUSSION

In this work, we formulate the quantization of one-particle classical system in terms of the variational approach. There are three requirements in the present derivation; 1) quantum fluctuation is expressed as stochastic noises of the Brownian motion, 2) the two stochastic flows from the past and the future are characterized by the maximum of the entropy associated with the number of combinatory processes connecting the fixed distributions, and 3) the action of the system is given by the same form of a classical ideal fluid, but contains the contribution from the internal energy. This fluid action can be cast into the action principle for the Schrödinger equation. In addition, the variational form permits us to define operators associated with physical observables in the ψ -representations via Noether's theorem. The usual definition of eigenvalues and corresponding eigenstates of the physical observables are consistently defined within this scheme. In short, the well-known hydrodynamic representation of the Schrödinger equation is reformulated within the framework of the stochastic variational method of the Nelson-Yasue approach.

The method developed here is easily applicable to the quantization of scalar fields. In this case, we consider that the velocity in the functional space is irrotational. As was pointed out by Takabayashi and Wallstrom, the usual hydrodynamic representation of the Schrödinger equation cannot treat the cases where the phase of wave functions becomes multi-valued, such as the eigenstates of the angular momentum unless introducing an additional condition which requires that the vortex number is quantized [13] [14]. However, in the field quantization, the vorticity in question refers to the flow in the functional space and nothing to do with multivaluedness of the phase of the wavefunction itself. For the case of scalar field the hydrodynamic equations in the functional space can be derived from the variation of Eq.(42) and the flow in the functional space can be taken always irrotational. In particular for $V = 0$, we obtain the correct energy eigenvalues as the stationary states without resort to the ψ -representation in functional space [8].

From the present study we may give rein our imagination for the possible origin of our quantum noise as the fluctuation of the space-time itself. In fact, the intensity of the noise for the field quantization is given uniquely by the universal constant $\kappa c = \hbar c$ as seen from Eqs.(45,46). This suggest that the field variables and the space-time structure were born simultaneously in such a way that quantum mechanics and relativity have the same origin

and are not to be defined separately. If this is the case, we would need to reconsider the meaning of quantization of gravity.

We have assumed that the stochasticity is characterized by the Gaussian white noise. This reminds us the well-known the central limit theorem and suggests that there may exist another hypermicro-stochastic process which is reduced to the Gaussian white noise only after taking the central limit theorem. This aspect seems to be consistent with the maximum entropy postulate which we have adopted, since the average of large micro-stochastic process should lead to certain statistical equilibrium.

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